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Constructing regular 2-starcompact spaces that are not strongly 2-star-Lindelöf

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Abstract

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It is shown that from any first-countable, zero-dimensional, locally compact, non-DFCC space X with the property that every nonempty open set has π-weight $\mathfrak{c}$ one can construct a pseudocompact space Y^+ that is not strongly 2-star-Lindelöf. Such a space Y^+ would therefore be

(1) 2-starcompact, T_3 but not strongly 2-starcompact;
(2) 2-star-Lindelöf, T_3 but not strongly 2-star-Lindelöf.

A space satisfying (1) has been constructed using CH and a Moore space with a σ-locally countable base satisfying (2) is known. The examples generated here are easier than these two spaces, require no set theory beyond ZFC and make the distinctions (1) and (2) simultaneously.

Keywords: Discrete finite chain condition, starcompact, pseudocompact, discrete countable chain condition, star-Lindelöf, π-base.

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Introduction

The purpose of this paper is to clarify the distinction between two classes of spaces discussed in [9]. With the definitions below, we will describe a scheme that generates pseudocompact spaces that are not strongly 2-star-Lindelöf. Such spaces will therefore be

(1) 2-starcompact, T_3 but not strongly 2-starcompact;
(2) 2-star-Lindelöf, T_3 but not strongly 2-star-Lindelöf.

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In [9], a space satisfying property (1) is constructed using CH and, in [4], a Moore space with a σ-locally countable base satisfying (2) is described. The examples constructed here will neither be developable nor have a σ-locally countable base. This is because pseudocompact Moore spaces are strongly 2-starcompact and pseudocompact σ-para-Lindelöf spaces (and hence pseudocompact spaces with a σ-locally countable base) are Lindelöf [3]. However, the construction will involve no set theory beyond ZFC, will be simpler than the example in [4] and will satisfy (1) and (2) simultaneously.

All spaces will be Hausdorff. Recall that a collection of nonempty open sets $\mathcal{P}$ is a π-base for a space X if for every nonempty open set U in X there is some $P \in \mathcal{P}$ with $P \subseteq U$. The π-weight of X, $\pi w(X)$, is the least cardinality of a π-base. If $B \subseteq X$ and $\mathcal{H}$ is a collection of subsets of X, then $\mathrm{st}^1(B, \mathcal{H}) = \bigcup \{H \in \mathcal{H}: H \cap B \neq \emptyset\}$ and, inductively, $\mathrm{st}^{n+1}(B, \mathcal{H}) = \mathrm{st}(\mathrm{st}^n(B, \mathcal{H}), \mathcal{H})$. For a positive integer n, we make the following definitions:

Definition. A space X is said to be *n-starcompact* (respectively *n-star-Lindelöf*) if for every open cover $\mathcal{U}$ of X, there is some finite (respectively countable) subset $\mathcal{V}$ of $\mathcal{U}$ such that $\mathrm{st}^n(\bigcup \mathcal{V}, \mathcal{U}) = X$.

Definition. A space X is said to be *strongly n-starcompact* (respectively *strongly n-star-Lindelöf*) if for every open cover $\mathcal{U}$ of X, there is some finite (respectively countable) subset B of X such that $\mathrm{st}^n(B, \mathcal{U}) = X$.

Definition. A space X is said to have the *discrete finite chain condition* (is *DFCC*) if every discrete collection of open sets is finite.

Proofs of facts stated about these properties can be found in [9]. It follows from the results on lightly compact spaces in [1] that, for $T_{3\frac{1}{2}}$ spaces, pseudocompactness and the discrete finite chain condition are equivalent conditions. For T_3 spaces, 2-starcompactness is equivalent to the DFCC and 2-star-Lindelöf is equivalent to the discrete countable chain condition (DCCC).

A similar argument and the notation used here can be found in [2], where Bell discusses first-countable, meta-Lindelöf pseudocompactifications.

The example

Let X satisfy (*) if it is a first-countable, zero-dimensional, locally compact, non-DFCC space with the property that every nonempty open set has π-weight $\mathfrak{c}$. We will show that from any space X satisfying (*) one can construct a pseudocompact space Y^+ that is not strongly 2-star-Lindelöf. Y^+ also turns out to be first-countable, zero-dimensional and locally compact.

To begin with, we need to ensure that spaces satisfying (*) actually exist. The topological sum of ω copies of a first-countable, zero-dimensional, compact space where every nonempty open set has π-weight $\mathfrak{c}$ will suffice; the product of ω copies of the lexicographically ordered Cantor square is such a space.

Let $Y=\bigcup\{X_\alpha: \alpha<\mathfrak{c}\}$, the union of $\mathfrak{c}$ many copies of a space X satisfying (*). Let $\mathscr{B}$ be an open base for Y such that every $B\in\mathscr{B}$ is a compact clopen subset of some X_α. In [8], Šapirovskiĭ proves that a compact T_2 space with countable tightness has a point-countable π-base (a more accessible proof of this fact can be found in [5]). It follows that every locally compact T_2, first-countable space X has the same property. (Let $\mathscr{U}$ be a collection of pairwise disjoint open subsets of X such that $\bigcup\mathscr{U}$ is dense in X and $\bar{U}$ is compact whenever $U\in\mathscr{U}$. Let $\mathscr{P}_U$ be a point-countable π-base for $\bar{U}$. Then $\mathscr{P}=\bigcup_{U\in\mathscr{U}}\{V\cap U: V\in\mathscr{P}_U\}$ is a point-countable π-base for X, since $\mathscr{U}$ is a pairwise disjoint collection.) That the sample space above has this property can easily be checked directly.

Let $\mathscr{P}$ be a point-countable π-base for Y with $\mathscr{P}\subseteq\mathscr{B}$ and $|\mathscr{P}|=\mathfrak{c}$. Observe that every nonempty $B\in\mathscr{B}$ contains $\mathfrak{c}$ many other elements of $\mathscr{P}$; after all, $\{P\in\mathscr{P}: P\subseteq B\}$ is a π-base for B and hence has cardinality $\mathfrak{c}$.

Let $D(\mathscr{P})=\{S: S$ is a sequence $\{S_n: n\in\omega\}$ of disjoint open sets from $\mathscr{P}$ that have no cluster point in $Y\}$, i.e., all sequences from $\mathscr{P}$ witnessing that Y is not DFCC. Note that $|D(\mathscr{P})|=\mathfrak{c}$ because $|\mathscr{P}|=\mathfrak{c}$. Enumerate $D(\mathscr{P})$ as $\{T^\alpha: \alpha<\mathfrak{c}\}$ in such a way that $\bigcup T^{\omega\cdot\alpha}\subseteq X_\alpha$ for all $\alpha<\mathfrak{c}$. Suppose for all $\beta<\alpha$ we have chosen $S^\beta\in D(\mathscr{P})$ inductively in such a way that $S^\beta_n\subseteq T^\beta_n$ for every n and for $\beta,\beta'<\alpha$, $S^\beta_n=S^{\beta'}_{n'}$ implies $\beta=\beta'$ and $n=n'$. Because there are $\mathfrak{c}$ many elements of $\mathscr{P}$ that are subsets of T^α_n, we may pick S^α_n so that $S^\alpha_n\in\mathscr{P}-\{S^\beta_m: \beta<\alpha, m\in\omega\}$ and $S^\alpha_n\subseteq T^\alpha_n$. Define $S^\alpha=\{S^\alpha_n: n\in\omega\}$, which satisfies the inductive hypothesis at the αth stage. So for each T^α, we have picked a new sequence S^α refining it.

Let $\mathscr{S}=\{S^\alpha: \alpha<\mathfrak{c}\}$ and $\mathscr{A}=\{S^{\omega\cdot\alpha}: \alpha<\mathfrak{c}\}$. By Zorn's lemma, let $\mathscr{S}'$ be a maximal "eventually disjoint" subfamily of $\mathscr{S}$ containing $\mathscr{A}$. By eventually disjoint, we mean that if $S\neq S'\in\mathscr{S}'$ there is some $n\in\omega$ such that $S_m\cap S'_{m'}=\emptyset$ whenever $m, m'\geqslant n$. By maximal, we mean that if $S\in\mathscr{S}$ there is some $S'\in\mathscr{S}'$ such that, for any $n\in\omega$, there are $m, m'\geqslant n$ with $S_m\cap S'_{m'}\neq\emptyset$.

Let $Y^+=Y\cup\mathscr{S}'$, $O_n(S)=\{S\}\cup\bigcup_{m\geqslant n}S_m$ for every $S\in\mathscr{S}'$, $n\in\omega$ and $\mathscr{O}=\{O_n(S): S\in\mathscr{S}', n\in\omega\}$. Let $\mathscr{B}'=\mathscr{B}\cup\mathscr{O}$. It is straightforward to check that $\mathscr{B}'$ is a base for a topology on Y^+, making it first-countable, locally compact, zero-dimensional and T_2.

To see that Y^+ is DFCC, suppose, for a contradiction, that $\mathscr{V}=\{V_n: n\in\omega\}$ were a discrete collection of nonempty open subsets of Y^+. As Y is dense and open in Y^+, for each n pick some $U_n\in\mathscr{P}$ such that $U_n\subseteq V_n$. Then, for some α, $\{U_n: n\in\omega\}=T^\alpha\in D(\mathscr{P})$. By the maximality of $\mathscr{S}'$, there is some $S\in\mathscr{S}'$ that is not eventually disjoint from T^α. Therefore $S\in Y^+$ is a cluster point of T^α and hence also a cluster point of $\mathscr{V}$, a contradiction. Since Y^+ is $T_{3\frac{1}{2}}$ and DFCC, it is therefore pseudocompact.

Finally, we show that the open cover $\mathscr{U}=\{X_\alpha: \alpha<\mathfrak{c}\}\cup\{O_1(S): S\in\mathscr{S}'\}$ witnesses that Y is not strongly 2-star-Lindelöf. $\mathscr{U}$ has the following properties:

(1) every point $y\in Y$ is contained in at most countably many elements from $\mathscr{U}$;

(2) if $\mathscr{V}\subseteq\mathscr{U}$ is countable, there is an $S\in\mathscr{S}'$ such that $O_1(S)\cap\bigcup\mathscr{V}=\emptyset$.

To see (1), $\mathcal{O}$ is point-countable, so $\mathcal{U}$ is too. To see (2), since each $O_n(S)$ meets at most countably many X_α, $A=\{\alpha<\mathfrak{c}: X_\alpha \cap \bigcup \mathcal{V} \neq \emptyset\}$ is countable. Because $\mathcal{A} \subseteq \mathcal{S}'$, whenever $\beta \notin A$ we have $O_1(S^{\omega \cdot \beta}) \cap \bigcup \mathcal{V} \subseteq X_\beta \cap \bigcup \mathcal{V} = \emptyset$.

Let $C \subseteq Y^+$ be countable. By (1), $\{U \in \mathcal{U}: U \cap C \neq \emptyset\}$ is countable. So, by (2), let $S \in \mathcal{S}'$ be such that $O_1(S) \cap \mathrm{st}^1(C, \mathcal{U}) = \emptyset$. But $O_1(S)$ is the only element of $\mathcal{U}$ containing S, so $S \notin \mathrm{st}^2(C, \mathcal{U})$, i.e., $\mathrm{st}^2(C, \mathcal{U}) \neq Y$. This completes the proof.

Remarks. (a) Just as in Bell's construction, Y^+ will be meta-Lindelöf if and only if the starting space X is.

(b) For each T_3 first-countable space X_0 there is a Moore space S with the property that S is separable, CCC, DCCC if and only if X_0 has the corresponding property. In addition, there is a continuous surjection f from S onto X_0. This "Moore space machine" is due to Reed and is described in [7]. Reed has kindly pointed out that, applying the Moore space machine to Y^+, we obtain a DCCC Moore space S that is not strongly 2-star-Lindelöf. S is DCCC because Y^+ is. S cannot be strongly 2-star-Lindelöf, for otherwise Y^+ would be the continuous image of a strongly 2-star-Lindelöf space, so would itself be strongly 2-star-Lindelöf. Therefore S is a Moore space satisfying condition (2) in the Introduction. Unfortunately, this S will not have a σ-locally countable base, unlike the example in [4].

It was mentioned in the Introduction that pseudocompact Moore spaces are strongly 2-starcompact. This is because pseudocompact Moore spaces are separable [6] and separable 2-starcompact spaces are strongly 2-starcompact. Furthermore, spaces with the countable chain condition are 1-star-Lindelöf. This leaves the following nagging question: although there cannot exist an example which is not strongly 2-star-Lindelöf, does there exist a pseudocompact CCC space that is not strongly 2-starcompact?

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